Quantizing the EM field.

Let us start with Maxwell's equations (units are cgs, but please add c, μ, ε as needed)

$$(1)\ \nabla\cdot E = 4\pi\rho \qquad (3)\ \nabla\cdot B = 0$$

$$(2)\ \nabla\times B = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial E}{\partial t} \qquad (4)\ \nabla\times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

Let us introduce the vector + the scalar potentials

$$\left.\begin{array}{l} B = \nabla\times\vec{A} \\ E = -\nabla\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\end{array}\right\} \quad \begin{array}{l} \nabla\cdot B = \nabla\cdot(\nabla\times\vec{A}) = 0 \\ \nabla\times E = -\nabla\times\nabla\varphi - \frac{1}{c}\nabla\times\frac{\partial\vec{A}}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times A = -\frac{1}{c}\frac{\partial B}{\partial t}\end{array}$$

Plugging these in, we automatically satisfy eqns. (3) + (4)

The remaining equations become:

$$\nabla\cdot E = -\nabla^2\varphi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot A) = \rho \quad\Rightarrow\quad \left[\frac{1}{c^2}\partial_t^2\varphi - \nabla^2\varphi\right] - \frac{1}{c}\partial_t\left[\frac{1}{c}\partial_t\varphi + \nabla\cdot\vec{A}\right] = \rho$$

$$\nabla\times B - \frac{1}{c}\frac{\partial E}{\partial t} = \nabla(\nabla\cdot A) - \nabla^2 A - \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}\right) = \frac{4\pi}{c} j \quad\Rightarrow\quad \left[\frac{1}{c^2}\partial_t^2 A - \nabla^2 A\right] + \nabla\left[\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot A\right] = \frac{4\pi}{c} j$$

Let me re-write these in a "nicer" notation

introduce: $\partial^\mu = \left(\frac{1}{c}\partial_t, -\vec{\nabla}\right) \qquad j^\mu = (c\rho, 4\pi\vec{j})$

$A^\mu = (\varphi, \vec{A})$

With these definitions, we can get the EM fields from

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

The inhomogenous eq. become

$$\boxed{\partial_\nu F^{\mu\nu} = \frac{1}{c} j^\mu}$$

Note continuity eq: $\partial_\mu\partial_\nu F^{\mu\nu} = \frac{1}{c}\partial_\mu j^\mu$

$= \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$

Gauge Symmetry:

The description in terms of A^μ is redundant $\Rightarrow$

we can see this redundancy by the gauge transformation

$$A^\mu \to A^\mu + \left[\frac{1}{c}\partial_t, -\vec{\nabla}\right]\Lambda \equiv A^\mu + \partial^\mu\Lambda$$

Gauge transform leaves the original $\vec{E}$ and $\vec{B}$ fields unchanged:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \longrightarrow \partial^\mu A^\nu + \cancel{\partial^\mu \partial^\nu \Lambda} - \partial^\nu A^\mu - \cancel{\partial^\nu \partial^\mu \Lambda} = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$

- This invariance holds for any $\Lambda(t, \vec{x})$.
- The invariance is "local" [to be distinguished from global symmetries]
- Local gauge invariances are fundamental symmetries of nature + play an important role for E-M, QCD, weak + strong forces...

[Maybe gravity?]

Why introduce gauge fields at all?

Gauge fields let us introduce coupling between charged particles and the EM fields in a particularly nice way:

Minimal substitution: $p \to p - \frac{e}{c} A$

This let's us modify Schrodinger's equation to include gauge fields

$$i\hbar\, \partial_t \psi = \frac{1}{2m}\left[-i\hbar \nabla\right]^2 \psi \quad \longrightarrow \quad i\hbar\, \partial_t \psi - e\phi\psi = \frac{1}{2m}\left[-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right]^2 \psi$$

$$i\hbar\, \partial_t \psi = H\psi = \left(\frac{1}{2m}\left[-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right]^2 + e\phi\right)\psi$$

[Schrodinger's eq. for charged particle moving in E-M field.

(1) Easy to check that we get Lorentz force

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi$$

Hamilton's EOM: (1) $\dot{p}_i = -\frac{\partial H}{\partial q_i} = +\frac{1}{2m}\frac{e}{c}\frac{\partial}{\partial q_i} A_j \left(p - \frac{e}{c}A\right)_j - e\frac{\partial}{\partial q_i}\phi$

(2) $\dot{q} = \frac{\partial H}{\partial p} = \frac{1}{m}\left(p - \frac{e}{c}\vec{A}\right)$

(2) $\Rightarrow$ $p = m\dot{q} + \frac{e}{c}A$

(1) $\Rightarrow$ $\dot{p} = m\ddot{q} + \underline{\underline{\frac{e}{c}\dot{A}}} = \frac{e}{c}\underline{\left(\frac{\partial}{\partial q_i} A_j\right)\cdot \dot{q}_j} - \underline{\underline{e\frac{\partial}{\partial q_i}}}\phi$

$$m\ddot{q} = \frac{e}{c}\,\underline{V \times \vec{B}} + \underline{\underline{e\vec{E}}} \qquad (\checkmark) \qquad E = e\left[-\nabla\phi - \frac{1}{c}\partial_t\vec{A}\right]$$

Verify me!

(2) If we wanted to write H in terms of B + E, and still get the Lorentz force correct the form of $\mathcal{H}$ would be much uglier [we would need to resort to time integrals?]

(3) Easy to check that $H \to H$ under the gauge transform $A^\mu \to A^\mu + \partial^\mu \Lambda$ $\psi \to \psi e^{ie\Lambda/\hbar c}$.

Quantizing the E-M field:

Let us figure out what are the physical degrees of freedom for the source-free case $j = \rho = 0$ i.e. for E-M field in vacuum.

(1) We need to fix the gauge $\Rightarrow$ choose the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$

(2) From $\nabla \cdot E = \rho = 0$ we obtain $\nabla \cdot E = -\nabla^2 \varphi - \frac{1}{c}\partial_t(\nabla \cdot A) = -\nabla^2 \varphi = 0$

$\Rightarrow$ this can be solved by setting $\boxed{\varphi = 0.}$

(3) The remaining eq. becomes:

$$\left[\frac{1}{c^2}\partial_t^2 A - \nabla^2 A\right] + \nabla\left[\cancelto{0}{\frac{1}{c}\frac{\partial \varphi}{\partial t}} + \cancelto{0}{\nabla \cdot A}\right] = \frac{4\pi}{c} j = 0$$

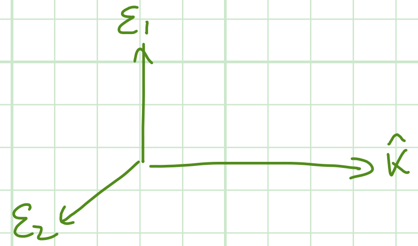
from gauge choice

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) A = \Box \vec{A} = 0$$

Maxwell's wave equation for the vector potential.

(4) solve $\Rightarrow$ $A(x,t) = \vec{\varepsilon}(k) e^{-i(\omega t - kx)}$ where $\omega = \pm c|k|$

and $\nabla \cdot A = 0 \Rightarrow \vec{\varepsilon} \perp \vec{k}$



(5) Since Maxwell's wave equation is linear, we can compose any solution of it via linear superpositions of waves:

$$A(x,t) = \frac{1}{\sqrt{V}} \sum_{k} \sum_{\lambda=1,2} \frac{\vec{\varepsilon}_\lambda(k)}{\sqrt{2\omega_k}} \left[a_\lambda(k) e^{-i(\omega_k t - k\cdot x)} + a_\lambda^*(k) e^{i(\omega_k t - k\cdot x)} \right]$$

def of FT

normalizatio factor added because We know answer

(6) let's compute the energy stored in the E-M field

$$H = \frac{1}{2}\int d^3x \left[E^2 + B^2\right]$$

where $E = -\frac{1}{c}\partial_t A$ and $B = \nabla \times A$

plugging in our generic solution and doing a bunch of Algebra like:

$$E = +\frac{i}{c}\frac{1}{\sqrt{V}}\sum_{k}\sum_{\lambda=\pm 1}\frac{\vec{\varepsilon}_\lambda(k)}{\sqrt{2}}\sqrt{\omega_k}\left[a_\lambda(k)\, e^{-i(\omega_k t - k\cdot x)} - a^*_\lambda(k)\, e^{i(\omega_k t - k\cdot x)}\right]$$

note almost the result from last time

We obtain:

$$H = \frac{1}{2}\sum_{k,\lambda}\left[a^*_\lambda(k)a_\lambda(k) + a_\lambda(k)a^*_\lambda(k)\right]\omega_k$$

(7) second quantize

$$a_\lambda(k) \rightarrow \sqrt{\hbar}\;\hat{a}_\lambda(k)$$

$$a^*_\lambda(k) \rightarrow \sqrt{\hbar}\;\hat{a}^+_\lambda(k)$$

and introduce the commutation relation: $[\hat{a}_\lambda(k), \hat{a}^+_{\lambda'}(k')] = \delta_{\lambda\lambda'}\delta_{kk'}$

$$H = \sum_{k\lambda}\hbar\omega_k\left[\hat{a}^+_\lambda(k)\,\hat{a}_\lambda(k) + \frac{1}{2}\right]$$

(8) $$E(x,t) = +\frac{i}{c}\frac{1}{\sqrt{V}}\sum_{k,\lambda}\vec{\varepsilon}_\lambda(k)\sqrt{\frac{\hbar\omega_k}{2}}\left[\hat{a}_\lambda(k)\, e^{-i(\omega_k t - k\cdot x)} - \hat{a}^+_\lambda(k)\, e^{i(\omega_k t - k\cdot x)}\right]$$

coupling phonons + photons, what about time?

last time $\Rightarrow$ we did not have the explicit time dependence in $\vec{E}(x,t)$ nor $\vec{P}(x,t)$. This is ok since in the schrodinger picture H should be independent of time. Hence we can pick our favorite time e.g. $t=0$ when evaluating H.

Explicitly we can get rid of the time dependent phases with a gauge transform ...

What about Bogoliubov transforms?

Last time we claimed that the correct problem to solve was

not $H\psi = \varepsilon\psi$ but instead $(\Sigma H)\psi = \varepsilon\psi$ where $\Sigma = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

or

$H\psi = \varepsilon\Sigma\psi$

Let us go through this carefully

(a) To diagonalize the Hamiltonian H we want a real linear transform on the Bosonic operators

$$\vec{b} = M\vec{\xi}$$

$$\vec{b}^{\dagger} = \vec{\xi}^{\dagger} M^T$$

explicitly this notation stands for:

$$\vec{b} = \begin{pmatrix} a_k \\ c_k \\ a^{\dagger}_{-k} \\ c^{\dagger}_{-k} \end{pmatrix} = M \begin{pmatrix} \xi_{1,k} \\ \xi_{2,k} \\ \xi^{\dagger}_{1,-k} \\ \xi^{\dagger}_{2,-k} \end{pmatrix}$$

where $\xi_{i,k}$ are the "new" operators.

With this transformation the Hamiltonian becomes:

$$\vec{b}^{\dagger} H \vec{b} = \vec{\xi}^{\dagger} M^T H M \vec{\xi} = \vec{\xi}^{\dagger} D \vec{\xi}$$

Where $D = \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \ddots \end{pmatrix}$ is the matrix of eigen values on the diagonal

Hence our goal is to find the Matrix M that satisfies the equation:

$$M^T H M = D$$

(b) we want the same commutation relations to hold for ξ's and b's

this imposes an extra condition on the matrix M:

$$M^T \Sigma M = \Sigma$$

To see the origin of this condition consider

$$(\vec{b}\,\vec{b}^{\dagger})^T - (\vec{b}^{\dagger})^T(\vec{b})^T = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = [M\vec{\xi}\,\vec{\xi}^{\dagger}M^T]^T - [\vec{\xi}^{\dagger}M^T]^T[M\vec{\xi}]^T$$

use the commutation conditions on b's

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \equiv \Sigma$$

$$= M[\vec{\xi}\,\vec{\xi}^{+}]^T \; M^T - M[\vec{\xi}^{+}]^T[\vec{\xi}]^T M^T$$

$$= M\left([\vec{\xi}\,\vec{\xi}^{+}]^T - [\vec{\xi}^{+}]^T[\vec{\xi}]^T \right) M^T \quad = M\Sigma M^T$$

by fact that commutation relations on ξ's should be same as those for b's.

(c) Making use of (a) + (b):

$$M\Sigma M^T = \Sigma \quad \Rightarrow \quad \Sigma M^T = M^{-1}\Sigma \quad \Rightarrow \Sigma^2 M^T = M^T = \Sigma M^{-1}\Sigma$$

$$M^T H M = D \quad \Rightarrow \quad \Sigma M^{-1}\Sigma H M = D \quad \Rightarrow \quad M^{-1}[\Sigma H]M = [\Sigma D]$$

This is an eigenvalue problem for the matrix $\tilde{H} = \Sigma H$ with the eigenvalues $\tilde{D} = \Sigma D$.

$$(\Sigma H)\Psi = \tilde{\varepsilon}\Psi \qquad \text{where} \qquad \varepsilon_i = \Sigma_{ii}\,\tilde{\varepsilon}_i$$

We can restate the problem as a generalized eigenvalue problem by multiplying both sides by Σ

$$H\Psi = \tilde{\varepsilon}\Sigma\Psi$$

Quantum description of polaritons, end game ...

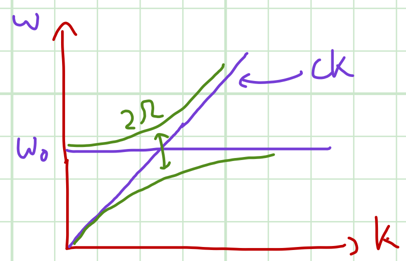
$$\begin{array}{c|cccc|}
 & a_K & c_K & a_{-K}^{+} & c_{-K}^{+} \\
c_K^{+} & \omega_K & -i\Omega & 0 & -i\Omega \\
c_K^{+} & i\Omega & \omega_0 & -i\Omega & 0 \\
a_{-K} & 0 & i\Omega & \omega_K & i\Omega \\
c_{-K} & i\Omega & 0 & -i\Omega & \omega_0
\end{array}
\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \omega \begin{pmatrix} \alpha \\ \beta \\ -\gamma \\ -\delta \end{pmatrix} \Bigg\} \Rightarrow$$

these signs are needed to ensure commutation relations between the new operators are satisfied.

⇒ I think OS. has an error in equation for ω, p. 351. $-4\Omega^2\omega_k^2 \Rightarrow -4\Omega^2\omega_k\omega_0$

$$\Rightarrow \mathrm{Det}\left[\mathcal{H} - \begin{pmatrix}\omega & & & \\ & \omega & & \\ & & -\omega & \\ & & & -\omega\end{pmatrix}\right] = \omega^4 - \omega^2(\omega_0^2+\omega_k^2) + \omega_0^2\omega_k^2 - 4\Omega^2\omega_0\omega_k = 0$$

Plot the roots:



$\Omega \to 0 \Rightarrow$

$$\omega^4 - \omega^2(\omega_0^2+\omega_k^2) + \omega_0^2\omega_k^2 = 0$$

$$\omega^2 = \frac{\omega_0^2+\omega_k^2 \pm \sqrt{(\omega_0^2+\omega_k^2)^2 - 4\omega_0^2\omega_k^2}}{2}$$

$$= \frac{\omega_0^2+\omega_k^2 \pm |\omega_0^2-\omega_k^2|}{2}$$

$$\omega = \{\omega_0, \omega_k\}$$

$k \to 0 \Rightarrow \omega^4 - \omega_0^2\omega^2 = 0$

$$\omega = \{0, \omega_0\}$$

$\frac{\Omega}{\omega_0}$ small: $\omega \to \omega_0 \pm \Omega$

$ck = \omega_0$ $\qquad \psi = \frac{1}{\sqrt{2}}(a_k \pm i c_k)$

Exciton-polaritons

Instead of optical phonons we have excitons ⇒ bound p-h pairs

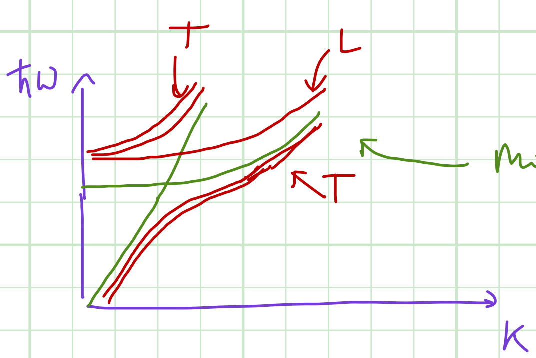
$$E_x = E_{gap} - R_{y_{ex}} + \frac{\hbar^2k^2}{2m_{ex}}$$

E_{gap}: gap between V + C bands

$R_{y_{ex}}$: binding Energy

$\frac{\hbar^2k^2}{2m_{ex}}$: COM KE

⇒ to proceed, we substitute E_x for $\hbar\omega_0$ in the phonon-polariton expressions



main modification is spectrum becomes curved at high k.